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Brownian Motion in a Rotating Flow

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The Langevin equations for a particle of an arbitrary shape and the correlation functions for the fluctuating forces, torques, or force-torque acting on the particle in a rotating flow are derived from the semimicroscopic level of coarse graining by using fluctuating hydrodynamics. In order to obtain the solution of the Navier–Stokes Langevin equation valid over the entire flow region, use is made of the method of matched asymptotic expansions in $(\Omega_f' a^2/\nu)^{1/2} \ll 1$. The cases of slow and rapid rotation are analyzed. It is shown that the fluctuation-dissipation theorems hold up to the order of $(\Omega_f' a^2/\nu)^{1/2}$ in both slow and rapid rotation, and that the diffusivity tensor depends on the angular velocity of the fluid and becomes anisotropic.

KEY WORDS: Langevin equation; Brownian motion; rotating flow; fluctuating hydrodynamics; fluctuation-dissipation theorem; method of matched asymptotic expansions.

1. INTRODUCTION

From the first successful theory of Brownian motion due to Einstein[1] there have been many studies of random motion of suspending particle in a quiescent fluid, which has been considered as a prime example of non-equilibrium phenomena. The random motion of a Brownian particle can be described by the generalized Langevin equation[2]

$$m \frac{dU_i(t)}{dt} = -\int_{-\infty}^{t} ds\, \zeta_{ij}(t-s)\, U_j(s) + F_i(t) \tag{1.1}$$

$$\langle F_i(t) \rangle = 0, \qquad \langle F_i(t)\, F_j(s) \rangle = 2k_B T \zeta_{ij}(|t-s|) \tag{1.2}$$

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where m is the mass of the particle and $U_i(t)$ its velocity, $\zeta_{ij}(t)$ is the time-dependent Stokes friction tensor, $F_i(t)$ is the fluctuating force due to the thermal agitation, and $\langle \cdot \rangle$ denotes the thermal equilibrium ensemble average, and the summation convention is used throughout this paper. The generalized Langevin equation (1.1) and the fluctuation-dissipation theorem (FD theorem) (1.2) are quite general and valid for any fluctuations in a system in thermal equilibrium, and can be derived on the grounds of statistical mechanics.[2,3] It is, however, difficult to compute $\zeta_{ij}(t)$ explicitly even for the spherical Brownian particle from the Liouville equation on first principles of statistical mechanics. An alternative way to obtain the explicit form of the FD theorem is to use linearized fluctuating hydrodynamics (LFHD). LFHD can be considered as a semimicroscopic theory in that the FD theorem is assumed on basis of the linearized Navier–Stokes Langevin equation.[4–9] The relevance of the LFHD has been clearly shown in the study of thermal fluctuations. In fact, the velocity autocorrelation for the Brownian particle can be expressed explicitly in terms of its friction coefficient which causes the long-time tail of the correlation function.[10–12] Many applications have been made for example, to treat particles of arbitrary shape,[13,14] many-particle systems,[15] polymers,[16] and wall effects.[17]

The extension of the theory of thermal fluctuations to the nonlinear regime has also been attempted. The theoretical basis for fluctuating hydrodynamics (FHD) far from equilibrium has been presented by Keizer[18] from the viewpoint of elementary molecular processes (see also Fox[19]), and it has been shown that the basic equations are governed by the Navier–Stokes Langevin equation. The thermal fluctuations in a fluid under the presence of a uniform steady velocity or temperature gradient have been examined by using the FHD[20–23] and it was found that the asymmetry in the correlation function of the density–density fluctuations in frequency space is in agreement with the experimental results. The Brownian motion in a fluid near the critical point of Rayleigh–Bénard convection has been examined by Lekkerkerker[24] Garisto and Mazur,[25] who found that the friction constant and diffusion coefficient for a spherical Brownian particle are proportinal to $\varepsilon^{1/2}$ and $\varepsilon^{-3/2}$ (divergent) as ε tends to zero, respectively, where $\varepsilon = (\mathrm{Ra}_c - \mathrm{Ra})/\mathrm{Ra}_c$, and Ra and Ra_c are the Rayleigh and critical Rayleigh numbers, respectively.

The above analyses for Brownian motion were carried out in the absence of macroscopic flow. It is interesting to consider the validity of the FD theorem when there exists a macroscopic flow. For a Brownian particle in a constant uniform flow, Kaneda[26,27] derived the nonlinear Langevin equation and obtained the FD theorem valid up to $O(R)$ by using the Navier–Stokes Langevin equation, where $R = aU_0/\nu$ and a is the characteristic particle length, U_0 is the particle velocity, and ν is the kinematic viscosity of the fluid. Hermans[28] used the Oseen Langevin equation, which is not consistent with Keizer's results, and it is known that the Oseen equation is not a correct approximation near the particle. Rubi and Bedeaux[29] recently studied Brownian motion in an elongational flow and found that to linear order of the penetration depth the $t^{-3/2}$ long-time tail of the velocity correlation function is modified and the anisotropic FD theorem obtained even for the spherical particle. Ryskin[30] considered Brownian motion in an ultracentrifuge and showed on phenomenological grounds that the diffusion coefficients depend on the square root of the rotation rate.

I shall consider the Langevin equations for a Brownian particle in a rotating flow with constant angular velocity Ω'_f, from the semimicroscopic viewpoint.[31] Since the fluid has an anisotropic nature due to the rotation, it would be expected that the Langevin equations are modified and the FD relations become anisotropic. I consider the zero-frequency limit of the correlation functions of the random forces and torques acting on the particle because it determines the diffusion coefficients. To see the effects of the macroscopic flow, we need to consider the nonlinear terms of the Navier–Stokes Langevin equation and thus to use the method of matched asymptotic expansions as studied by Kaneda.[26,27] I shall present the analysis mainly for the slow rotation case, i.e., $aU_0/\nu \ll (a^2\Omega'_f/\nu)^{1/2} \ll 1$, but for the fast rotating case $[U_0/\Omega'_f a \ll (a^2\Omega'_f/\nu)^{1/2} \ll 1]$ a brief analysis is shown. It is found that to first order in $(a^2\Omega'_f/\nu)^{1/2}$ the rotation causes the friction tensor to have nondiagonal components and the correlation functions of the fluctuating forces or torques to be anisotropic. In the following sections I will analyze these problems by using matched asymptotic expansions and Kaneda's method[26,27] of using correlation functions, to obtain clearly boundary conditions for the fluctuating fields that have not been explicitly presented in previous work.

2. BASIC EQUATIONS

Consider particles suspended in an infinite region of an incompressible fluctuating fluid which is undergoing rigid rotation with constant angular velocity $\mathbf{\Omega}'_f$. We assume that the suspension is so dilute that the interaction between the particles can be neglected. A particle of an arbitrary shape with characteristic length a and mass m is assumed to be translated with speed $\mathbf{U}'_B$ relative to the systematic unperturbed flow and to be rotating with angular velocity $\mathbf{\Omega}'_B$. The origin of the Cartesian coordinate system is chosen to be fixed to the particle and for the fluid to rotate about the x_2 axis. In this coordinate system, the unperturbed flow is expressed as

$\mathbf{v}'_0 = -\mathbf{U}'_B + \mathbf{\Omega}'_f \times \mathbf{r}'$ [or $= -\mathbf{U}'_B + \mathbf{C}' \cdot \mathbf{r}'$, $C'_{ij} = |\mathbf{\Omega}'_f| (\delta_{i1}\delta_{j3} - \delta_{i3}\delta_{j1})$]. The motion of the fluid is assumed to be described by the Navier–Stokes Langevin equations

$$\rho \left[\frac{\partial \mathbf{v}'}{\partial t'} + (\mathbf{v}' \cdot \nabla')\mathbf{v}' \right] = \nabla' \cdot (\boldsymbol{\tau}' + \boldsymbol{\sigma}') - \rho \frac{d\mathbf{U}'_B}{dt'}, \qquad \nabla' \cdot \mathbf{v}' = 0 \tag{2.1}$$

$$\tau'_{ij} = -p'\delta_{ij} + \mu 2e'_{ij}, \qquad 2e'_{ij} \equiv \left(\frac{\partial v'_i}{\partial x'_j} + \frac{\partial v'_j}{\partial x'_i} \right) \tag{2.2}$$

where τ'_{ij} is the stress tensor due to the velocity field $\mathbf{v}'$ and the pressure p'. It is assumed that the random stress tensor σ'_{ij} due to the thermal agitation has the stochastic properties

$$\langle \sigma'_{ij}(\mathbf{r}', t') \rangle = 0 \tag{2.3}$$

$$\langle \sigma'_{ij}(\mathbf{r}'_1, t'_1)\, \sigma'_{kl}(\mathbf{r}'_2, t'_2) \rangle = 2k_{\mathrm{B}} T \mu \gamma_{ijkl} \delta(\mathbf{r}'_1 - \mathbf{r}'_2)\, \delta(t'_1 - t'_2) \tag{2.4}$$

$$\gamma_{ijkl} = \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \tfrac{2}{3}\delta_{ij}\delta_{kl} \tag{2.5}$$

The temperature T is assumed to be constant throughout the fluid and $\langle \cdot \rangle$ denotes the "local" equilibrium ensemble average with $\mathbf{U}'_B$, $\mathbf{\Omega}'_B$, and $\mathbf{C}'$ fixed. The velocity field $\mathbf{v}'(\mathbf{r}', t')$ satisfies the following stick boundary condition on the surface S_p of the particle

$$\mathbf{v}'(\mathbf{r}', t') = \mathbf{\Omega}'_B \times \mathbf{r}', \qquad \mathbf{r}' \text{ on } S_p \tag{2.6}$$

The motion of the particle is governed by

$$m \frac{d\mathbf{U}'_B}{dt'} = \mathbf{F}' \equiv \int_{S_p} (\boldsymbol{\tau}' + \boldsymbol{\sigma}') \cdot d\mathbf{S}' \tag{2.7}$$

$$\mathbf{J}' \cdot \frac{d\mathbf{\Omega}'_B}{dt'} = \mathbf{M}' \equiv \int_{S_p} \mathbf{r}' \times (\boldsymbol{\tau}' + \boldsymbol{\sigma}') \cdot d\mathbf{S}' \tag{2.8}$$

where $\mathbf{J}'$ is the inertia tensor of the particle and $d\mathbf{S}'$ the area segment vector taken along the outward normal. If we choose the characteristic length, time, and velocity as a, $\lambda t_0 = a^2/\nu$, and $U_0 = (k_{\mathrm{B}} T/m)^{1/2}$, respectively, then the normalization yields the equations and boundary conditions as follows:

$$\lambda \frac{\partial \mathbf{v}}{\partial t} + R(\mathbf{v} \cdot \nabla)\mathbf{v} = \nabla \cdot (\boldsymbol{\tau} + \boldsymbol{\sigma}) - \lambda \frac{d\mathbf{U}_B}{dt}, \qquad \nabla \cdot \mathbf{v} = 0 \tag{2.9}$$

$$\tau_{ij} = -p\delta_{ij} + 2e_{ij} \tag{2.10}$$

$$\langle \sigma_{ij}(\mathbf{r}, t) \rangle = 0$$

$$\langle \sigma_{ij}(\mathbf{r}_1, t_1)\, \sigma_{kl}(\mathbf{r}_2, t_2) \rangle = \gamma_{ijkl} \delta(\mathbf{r}_1 - \mathbf{r}_2)\, \delta(t_1 - t_2) \tag{2.11}$$

and

$$\mathbf{v}(\mathbf{r}, t) = \boldsymbol{\Omega}_B \times \mathbf{r}, \qquad \mathbf{r} \text{ on } S_p \tag{2.12}$$

where $\lambda = \rho/\rho_B$ is the Lorentz parameter, ρ_B is the density of the particle, and $R = aU_0/\nu$ the translational particle Reynolds number. For later use we define here other dimensionless parameters: the rotational particle Reynolds number $R_\Omega = a^2\Omega_0/\nu$, the rotational Reynolds number $R_\kappa = R\kappa = a^2\Omega_f'/\nu$, and the dimensionless angular velocity $\kappa = \Omega_f' a/U_0$.

Let us write

$$\mathbf{v} = \bar{\mathbf{v}} + \tilde{\mathbf{v}}, \qquad p = \bar{p} + \tilde{p} \tag{2.13}$$

where for simplicity we use the notation $\bar{A}$ for $\langle A \rangle$. The field $(\bar{\mathbf{v}}, \bar{p})$ is a systematic field satisfying

$$\lambda \frac{\partial \bar{\mathbf{v}}}{\partial t} + R[(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} + \langle (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} \rangle] = \nabla \cdot \bar{\boldsymbol{\tau}} - \lambda \frac{d\mathbf{U}_B}{dt}, \qquad \nabla \cdot \bar{\mathbf{v}} = 0 \tag{2.14}$$

and the boundary conditions

$$\begin{aligned} \bar{\mathbf{v}} &= \boldsymbol{\Omega}_B \times \mathbf{r}, \qquad \mathbf{r} \text{ on } S_p \\ &\rightarrow -\mathbf{U}_B + \kappa \mathbf{C} \cdot \mathbf{r} \qquad \text{as} \quad |\mathbf{r}| \rightarrow \infty \end{aligned} \tag{2.15}$$

and the fluctuating field $(\tilde{\mathbf{v}}, p)$ satisfies

$$\lambda \frac{\partial \tilde{\mathbf{v}}}{\partial t} + R[(\bar{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} + (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} - \langle (\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}} \rangle] - \nabla \cdot \tilde{\boldsymbol{\tau}} = \nabla \cdot \tilde{\boldsymbol{\sigma}},$$

$$\nabla \cdot \tilde{\mathbf{v}} = 0 \tag{2.16}$$

$$\tilde{\mathbf{v}} = 0, \qquad \mathbf{r} \text{ on } S_p \tag{2.17}$$

The boundary conditions for $\tilde{\mathbf{v}}$ at infinity are not yet specified. The equations of motion of the particle are written as

$$\frac{d\mathbf{U}_B}{dt} = \bar{\mathbf{F}} + \tilde{\mathbf{F}}, \qquad \mathbf{J} \cdot \frac{d\boldsymbol{\Omega}_B}{dt} = \bar{\mathbf{M}} + \tilde{\mathbf{M}} \tag{2.18}$$

$$\bar{\mathbf{F}} \equiv \int_{S_p} \bar{\boldsymbol{\tau}} \cdot d\mathbf{S}, \qquad \bar{\mathbf{M}} \equiv \int_{S_p} \mathbf{r} \times \bar{\boldsymbol{\tau}} \cdot d\mathbf{S} \tag{2.19}$$

$$\tilde{\mathbf{F}} \equiv \int_{S_p} (\tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\sigma}}) \cdot d\mathbf{S}, \qquad \tilde{\mathbf{M}} \equiv \int_{S_p} \mathbf{r} \times (\tilde{\boldsymbol{\tau}} + \tilde{\boldsymbol{\sigma}}) \cdot d\mathbf{S} \tag{2.20}$$

where $(\bar{\mathbf{F}}, \bar{\mathbf{M}})$ and $(\tilde{\mathbf{F}}, \tilde{\mathbf{M}})$ are the systematic force and torque, and the fluctuating force and torque acting on the particle, respectively. We assume that parameters R, R_Ω, and R_κ are not only small, but more restrictive, i.e., they satisfy the conditions

$$R \ll 1, \qquad R_\Omega \ll 1, \qquad R_\kappa \ll 1 \tag{2.21}$$

and

$$R \ll R_\kappa^{1/2} \ll 1, \qquad \kappa \geqslant O(1) \tag{2.22}$$

The assumption $R \ll R_\kappa^{1/2}$ implies that the effect of inertia due to fluid rotation dominates that due to translation and rotation of the particle, while $R_\kappa^{1/2} \ll 1$ means slow rotation of the unperturbed flow.

3. FORMULATION BASED ON THE CORRELATION FUNCTIONS

We assume $\lambda \ll 1$, so that the time derivatives of the velocity fields can be neglected, which corresponds to the zero-frequency limit, and also neglect the term $R(\tilde{\mathbf{v}} \cdot \nabla)\tilde{\mathbf{v}}$ in (2.14) and (2.16) under the assumption of (2.22) (Appendix A). Then we obtain the equations and boundary conditions for the systematic field $(\bar{\mathbf{v}}, \bar{p})$ as

$$R(\bar{\mathbf{v}} \cdot \nabla)\bar{\mathbf{v}} = -\nabla\bar{p} + \Delta\bar{\mathbf{v}}, \qquad \nabla \cdot \bar{\mathbf{v}} = 0 \tag{3.1}$$

$$\begin{aligned} \bar{\mathbf{v}} &= \mathbf{\Omega}_B \times \mathbf{r}, \qquad \mathbf{r} \text{ on } S_p \\ &\to -\mathbf{U}_B + \kappa \mathbf{C} \cdot \mathbf{r} \qquad \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{3.2}$$

Since, from (3.1), $(\bar{\mathbf{v}}, \bar{p})$ is steady and $\tilde{\sigma}_{ij}(\mathbf{r}, t)$ is a statistically stationary process, $(\tilde{\mathbf{v}}, \tilde{p})$ is also a stationary process, so that correlation functions $\langle \tilde{v}_i(\mathbf{r}, t)\, \tilde{\sigma}_{kl}(\mathbf{x}, t') \rangle$, etc., are functions of $t - t'$. Hereafter we shall consider the systematic force and correlation function of the random forces for simplicity, and the torques or the interaction between the forces and torques will be shown if needed or at the final stage. Multiplying (2.16) and (2.17) by $\tilde{\sigma}_{kl}(\mathbf{r}, t')$, taking the ensemble average with $\mathbf{U}_B$, $\mathbf{\Omega}_B$, $\mathbf{C}$ fixed, and integrating over $t - t'$, we have

$$\begin{aligned} &\left\{ R\left(\delta_{ij}\bar{\mathbf{v}} \cdot \nabla + \frac{\partial \bar{v}_i}{\partial x_j}\right) - \delta_{ij}\Delta \right\} \langle \tilde{v}_j(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle + \frac{\partial}{\partial x_i} \langle \tilde{p}(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \\ &\quad = \frac{\partial}{\partial x_j} \langle \tilde{\sigma}_{ij}(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \end{aligned} \tag{3.3}$$

$$\frac{\partial}{\partial x_i} \langle \tilde{v}_i(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle = 0$$

$$\begin{aligned} \langle \tilde{v}_i(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle &= 0, \qquad \mathbf{x} \text{ on } S_p \\ &\to 0 \qquad \text{as} \quad |\mathbf{x} - \mathbf{r}| \to \infty \end{aligned} \tag{3.4}$$

Time integration with respect to $t-t'$ is not shown, for simplicity, until the final stage. Similarly multiplying (2.16) and (2.17) by $\tilde{F}_l(t')$, we obtain

$$\left\{R\left(\delta_{ij}\bar{\mathbf{v}}\cdot\nabla+\frac{\partial\bar{v}_i}{\partial x_j}\right)-\delta_{ij}\Delta\right\}\langle\tilde{v}_j(\mathbf{x})\tilde{F}_l\rangle+\frac{\partial}{\partial x_i}\langle\tilde{p}(\mathbf{x})\tilde{F}_l\rangle$$
$$=\frac{\partial}{\partial x_j}\langle\tilde{\sigma}_{ij}(\mathbf{x})\tilde{F}_l\rangle \tag{3.5}$$

$$\frac{\partial}{\partial x_i}\langle\tilde{v}_i(\mathbf{x})\tilde{F}_l\rangle=0$$

$$\langle\tilde{v}_i(\mathbf{x})\tilde{F}_l\rangle=0, \qquad \mathbf{x}\text{ on } S_p$$
$$\to 0, \qquad \text{as} \quad |\mathbf{x}|\to\infty \tag{3.6}$$

We now consider the correlation function of the fluctuating forces acting on the particle,

$$Y_{ij}=\int_{-\infty}^{\infty}\langle\tilde{F}_i(t)\,\tilde{F}_j(t')\rangle\,d(t-t')=\langle\tilde{F}_i\tilde{F}_j\rangle \tag{3.7}$$

which can be written by using (2.20) as

$$=\left\langle\int_{S_p}\{\tilde{\tau}_{ik}(\mathbf{x})+\tilde{\sigma}_{ik}(\mathbf{x})\}\,dS_k(\mathbf{x})\tilde{F}_j\right\rangle$$
$$=\int_{S_p}\{\langle\tilde{\tau}_{ik}(\mathbf{x})\tilde{F}_j\rangle+\langle\tilde{\sigma}_{ik}(\mathbf{x})\tilde{F}_j\rangle\}\,dS_k(\mathbf{x}) \tag{3.8}$$

If we know $\langle\tilde{\sigma}_{ik}(\mathbf{x})\tilde{F}_j\rangle$, then by solving (3.5) with (3.6) we can obtain the correlation function (3.7). To obtain the field $\langle\tilde{\sigma}_{ik}(\mathbf{x})\tilde{F}_j\rangle$, we define a field $(q^*_{im}(\mathbf{x}), p^*_m(\mathbf{x}))'$ adjoint to the problem (3.3) with a vanishing right-hand side, as follows:

$$-\left\{R\left(\delta_{ij}\bar{\mathbf{v}}\cdot\nabla-\frac{\partial\bar{v}_j}{\partial x_i}\right)+\delta_{ij}\Delta\right\}q^*_{jm}(\mathbf{x})+\frac{\partial}{\partial x_i}p^*_m(\mathbf{x})=0, \quad \frac{\partial}{\partial x_i}q^*_{im}(\mathbf{x})=0 \tag{3.9}$$

$$q^*_{im}(\mathbf{x})=\delta_{im}, \qquad \mathbf{x}\text{ on } S_p$$
$$\to 0 \qquad \text{as} \quad |\mathbf{x}|\to\infty \tag{3.10}$$

and the tensor for later use is

$$\theta^*_{ijm}\equiv -p^*_m\delta_{ij}+\frac{\partial}{\partial x_j}q^*_{im}+\frac{\partial}{\partial x_i}q^*_{jm} \tag{3.11}$$

Consider the identity

$$0 = q_{im}^{*}(\mathbf{x}) \left\{ \left[R\left(\delta_{ij} \bar{\mathbf{v}} \cdot \nabla + \frac{\partial \bar{v}_i}{\partial x_j} \right) - \delta_{ij} \Delta \right] \langle \tilde{v}_j(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \right.$$

$$\left. + \frac{\partial}{\partial x_i} \langle \tilde{p}(\mathbf{x}) \, \sigma_{kl}(\mathbf{r}) \rangle - \frac{\partial}{\partial x_j} \langle \tilde{\sigma}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \right\}$$

$$- \langle \tilde{v}_i(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \left\{ - \left[R\left(\delta_{ij} \bar{\mathbf{v}} \cdot \nabla - \frac{\partial \bar{v}_j}{\partial x_i} \right) + \delta_{ij} \Delta \right] q_{jm}^{*}(\mathbf{x}) + \frac{\partial}{\partial x_i} p_m^{*}(\mathbf{x}) \right\} \tag{3.12}$$

Using the relation[32]

$$\left[\frac{\partial}{\partial x_j} q_{im}^{*}(\mathbf{x}) \right] \tilde{\tau}_{ij}(\mathbf{x}) = \theta_{ijm}^{*}(\mathbf{x}) \left[\frac{\partial}{\partial x_j} \tilde{v}_i(\mathbf{x}) \right] \tag{3.13}$$

yields

$$0 = \frac{\partial}{\partial x_j} \{ q_{im}^{*}(\mathbf{x}) [\langle \tilde{\tau}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle + \langle \tilde{\sigma}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle]$$

$$- \langle \tilde{v}_i(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, \theta_{ijm}^{*}(\mathbf{x}) - R q_{im}^{*}(\mathbf{x}) \langle \tilde{v}_i(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, \bar{v}_j(\mathbf{x}) \}$$

$$- \left[\frac{\partial}{\partial x_j} q_{im}^{*}(\mathbf{x}) \right] \langle \tilde{\sigma}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle$$

Integrating this over the volume Γ bounded externally by the spherical surface S_A with radius A and by the surface S_p of the particle, and using Gauss' theorem and the boundary conditions (3.4) and (3.10), we have

$$\int_{S_p} [\langle \tilde{\tau}_{mj}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle + \langle \tilde{\sigma}_{mj}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle] \, dS_j(\mathbf{x})$$

$$= \int_{S_A} \{ q_{im}^{*}(\mathbf{x}) [\langle \tilde{\tau}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle + \langle \tilde{\sigma}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle]$$

$$- \langle \tilde{v}_i(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, \theta_{ijm}^{*}(\mathbf{x})$$

$$- R q_{im}^{*}(\mathbf{x}) \langle \tilde{v}_i(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, \bar{v}_j(\mathbf{x}) \} \, dS_j(\mathbf{x})$$

$$- \int_{\Gamma} \left[\frac{\partial}{\partial x_j} q_{im}^{*}(\mathbf{x}) \right] \langle \tilde{\sigma}_{ij}(\mathbf{x}) \, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, d\mathbf{x} \tag{3.14}$$

The left-hand side of (3.14) can be written as

$$\left\langle \left\{ \int_{S_p} [\tilde{\tau}_{mj}(\mathbf{x}) + \tilde{\sigma}_{mj}(\mathbf{x})] \, dS_j(\mathbf{x}) \right\} \tilde{\sigma}_{kl}(\mathbf{r}) \right\rangle = \langle \tilde{F}_m \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \tag{3.15}$$

On the other hand, from (2.11) we have

$$\langle \tilde{\sigma}_{ij}(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle = \int_{-\infty}^{\infty} \langle \tilde{\sigma}_{ij}(\mathbf{x}, t)\, \tilde{\sigma}_{kl}(\mathbf{r}, t') \rangle \, d(t-t') = \gamma_{ijkl}\, \delta(\mathbf{x}-\mathbf{r}) \tag{3.16}$$

and from (2.5) the last term of the right-hand side of (3.14) is

$$\int_{\Gamma} \left[\frac{\partial}{\partial x_j} q^*_{im}(\mathbf{x}) \right] \langle \tilde{\sigma}_{ij}(\mathbf{x})\, \tilde{\sigma}_{kl}(\mathbf{r}) \rangle \, d\mathbf{x} = \frac{\partial}{\partial r_k} q^*_{lm}(\mathbf{r}) + \frac{\partial}{\partial r_l} q^*_{km}(\mathbf{r}) \tag{3.17}$$

Assuming that the outer surface integral vanishes as $A \to \infty$ (Appendix B), we obtain

$$\langle \tilde{\sigma}_{kl}(\mathbf{r}) \tilde{F}_m \rangle = -\left[\frac{\partial}{\partial r_k} q^*_{lm}(\mathbf{r}) + \frac{\partial}{\partial r_l} q^*_{km}(\mathbf{r}) \right] \tag{3.18}$$

Thus, the field $\langle \sigma_{kl}(\mathbf{r}) \tilde{F}_m \rangle$ is expressed by the adjoint field $(q^*_{im}(\mathbf{x}),\ p^*_m(\mathbf{x}))^t$.

4. THE SYSTEMATIC FORCE AND TORQUE

We introduce perturbations $\mathbf{q}$ and p defined by

$$\begin{aligned} \bar{\mathbf{v}} &= \kappa \mathbf{C} \cdot \mathbf{r} - \mathbf{U}_B + \mathbf{q} \\ \bar{p} &= -\tfrac{1}{2} R\kappa^2 (x_1^2 + x_3^2) - R\kappa \mathbf{r} \cdot \mathbf{C} \cdot \mathbf{U}_B + p \end{aligned} \tag{4.1}$$

Then the equations and boundary conditions for $(\mathbf{q}, p)$ become

$$R(\kappa \mathbf{C} \cdot \mathbf{r} - \mathbf{U}_B + \mathbf{q}) \cdot \nabla \mathbf{q} + R\kappa \mathbf{C} \cdot \mathbf{q} = -\nabla p + \Delta \mathbf{q}, \qquad \nabla \cdot \mathbf{q} = 0 \tag{4.2}$$

$$\begin{aligned} \mathbf{q} &= -\kappa \mathbf{C} \cdot \mathbf{r} + \mathbf{U}_B + \mathbf{\Omega}_B \times \mathbf{r}, \qquad \mathbf{r} \text{ on } S_p \\ &\to 0 \qquad \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{4.3}$$

If we carry out the regular perturbation method in R, the zeroth-order solution is given by the Stokes solution, which has the asymptotic forms of $\mathbf{q}_0 = O(r^{-1})$ and $p_0 = O(r^{-2})$ for large r. However, this approximation is not valid over the entire flow region, because the neglected terms (nonlinear terms) are comparable with the viscous ones at the distance $r_* \equiv R_\kappa^{-1/2} \gg 1$. The zeroth-order solution is valid only over the region $r < r_*$. Therefore, it is necessary to consider the different expansions valid over the inner region $(r < r_*)$ and the outer region $(r > r_*)$, respectively. This method of expansions is called the method of matched asymptotic expansions.[33] In this paper, we use this method to obtain the solution up to $O(R_\kappa^{1/2})$ which is uniformly valid over the entire flow region.[34]

4.1. Inner Expansions

Under the assumption (2.22), the inner expansions are of the form

$$\mathbf{q} = \mathbf{q}_0(\mathbf{r}) + R_\kappa^{1/2}\mathbf{q}_1(\mathbf{r}) + o(R_\kappa^{1/2})$$
$$p = p_0(\mathbf{r}) + R_\kappa^{1/2} p_1(\mathbf{r}) + o(R_\kappa^{1/2}) \tag{4.4}$$

and the systematic force and torque are also expanded as

$$\bar{\mathbf{F}} = \bar{\mathbf{F}}_0 + R_\kappa^{1/2}\bar{\mathbf{F}}_1 + o(R_\kappa^{1/2})$$
$$\bar{\mathbf{M}} = \bar{\mathbf{M}}_0 + R_\kappa^{1/2}\bar{\mathbf{M}}_1 + o(R_\kappa^{1/2}) \tag{4.5}$$

where $\bar{\mathbf{F}}_0$ and $\bar{\mathbf{M}}_0$ are the force and torque due to $(\mathbf{q}_0,\ p_0)$, and $\bar{\mathbf{F}}_1$ and $\bar{\mathbf{M}}_1$ the force and torque due to $(\mathbf{q}_1, p_1)$, respectively. Upon substituting (4.4) into (4.2) and (4.3) and equating terms in R_κ^0, one obtains

$$\Delta\mathbf{q}_0 - \nabla p_0 = 0, \qquad \nabla\cdot\mathbf{q}_0 = 0 \tag{4.6}$$

$$\mathbf{q}_0 = -\kappa\mathbf{C}\cdot\mathbf{r} + \mathbf{U}_B + \mathbf{\Omega}_B\times\mathbf{r}, \qquad \mathbf{r} \text{ on } S_p$$
$$\to 0 \qquad \text{as} \quad |\mathbf{r}|\to\infty \tag{4.7}$$

Likewise, equating terms in $R_\kappa^{1/2}$,

$$\Delta\mathbf{q}_1 - \nabla p_1 = 0, \qquad \nabla\cdot\mathbf{q}_1 = 0 \tag{4.8}$$

$$\mathbf{q}_1 = 0 \qquad \mathbf{r} \text{ on } S_p \tag{4.9}$$

The boundary conditions for the field $(\mathbf{q}_1, p_1)$ as $r\to\infty$ are derived by the matching procedure.

4.2. Outer Expansions

The dimensionless outer variable $\tilde{\mathbf{r}}$ is defined as $\tilde{\mathbf{r}} = R_\kappa^{1/2}\mathbf{r}$. The outer expansions are

$$\mathbf{q} = R_\kappa^{1/2}\mathbf{Q}_1(\tilde{\mathbf{r}}) + o(R_\kappa^{1/2})$$
$$p = R_\kappa P_1(\tilde{\mathbf{r}}) + o(R_\kappa) \tag{4.10}$$

If the operators ∇ in (4.2) are rewritten in terms of the outer variable and the outer expansions (4.10) are substituted into the resulting equations, it is found that, to the lowest order, $(\mathbf{Q}_1, P_1)$ must satisfy

$$(\mathbf{C}\cdot\tilde{\mathbf{r}})\cdot\tilde{\nabla}\mathbf{Q}_1 + \mathbf{C}\cdot\mathbf{Q}_1 = -\tilde{\nabla}P_1 + \tilde{\Delta}\mathbf{Q}_1, \qquad \tilde{\nabla}\cdot\mathbf{Q}_1 = 0 \tag{4.11}$$

$$\mathbf{Q}(\tilde{\mathbf{r}}) \to 0 \qquad \text{as} \quad |\tilde{\mathbf{r}}|\to\infty \tag{4.12}$$

In addition, there are matching conditions at $\tilde{r}\to 0$.

4.3. Zeroth-Order Inner Approximation

The solution of (4.6) and (4.7) is clearly the Stokes solution of the problem. In our analysis, we require the knowledge of the Stokes field at great distances from the particle. The asymptotic forms for $r \to \infty$ are

$$\begin{aligned} \mathbf{q}_0(\mathbf{r}) &= -\mathbf{s}(\mathbf{r}) \cdot \bar{\mathbf{F}}_0 - [\nabla \mathbf{s}(\mathbf{r})] : \bar{\mathbf{B}}_0 + O(r^{-3}) \\ p_0(\mathbf{r}) &= -\mathbf{t}(\mathbf{r}) \cdot \bar{\mathbf{F}}_0 - [\nabla \mathbf{t}(\mathbf{r})] : \bar{\mathbf{B}}_0 + O(r^{-4}) \end{aligned} \tag{4.13}$$

where

$$s_{ij}(\mathbf{r}) = \frac{1}{8\pi r}\left(\delta_{ij} + \frac{r_i r_j}{r^2}\right), \qquad t_j(\mathbf{r}) = \frac{1}{4\pi r^2}\frac{r_j}{r}$$

$$(\bar{\mathbf{B}}_0)_{ij} - (\bar{\mathbf{B}}_0)_{ji} = \varepsilon_{ijk}(\bar{M}_0)_k \tag{4.14}$$

with $\bar{\mathbf{F}}_0$ and $\bar{\mathbf{M}}_0$ as the dimensionless Stokes force and torque.

4.4. First-Order Outer Approximation

Expressing the Stokes solution $(\mathbf{q}_0, p_0)$ in terms of the outer variable, we obtain

$$\begin{aligned} \mathbf{q}_0(\tilde{\mathbf{r}}) &= -R_\kappa^{1/2} \mathbf{s}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0 + O(R_\kappa) \\ p_0(\tilde{\mathbf{r}}) &= -R_\kappa \mathbf{t}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0 + O(R_\kappa^{3/2}) \end{aligned} \tag{4.15}$$

Hence, requirements for $(\mathbf{Q}_1, P_1)$ to be properly matched with the inner expansions are

$$\mathbf{Q}_1(\tilde{\mathbf{r}}) = -\mathbf{s}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0, \qquad P_1(\tilde{\mathbf{r}}) = -\mathbf{t}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0, \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to 0 \tag{4.16}$$

The solution of (4.11) subject to (4.2) and (4.16) is given by

$$\mathbf{Q}_1(\tilde{\mathbf{r}}) = -\mathbf{G}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0, \qquad P_1(\tilde{\mathbf{r}}) = -\mathbf{T}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0 \tag{4.17}$$

where the second-rank tensor $\mathbf{G}(\tilde{\mathbf{r}})$ and the vector $\mathbf{T}(\tilde{\mathbf{r}})$ satisfy

$$(\mathbf{C} \cdot \tilde{\mathbf{r}}) \cdot \tilde{\nabla}\mathbf{G} + \mathbf{C} \cdot \mathbf{G} = -\tilde{\nabla}\mathbf{T} + \tilde{\Delta}\mathbf{G} + \mathbf{I}\delta(\tilde{\mathbf{r}}), \qquad \tilde{\nabla} \cdot \mathbf{G} = 0 \tag{4.18}$$

$$\mathbf{G}(\tilde{\mathbf{r}}) \to 0 \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to \infty \tag{4.19}$$

It is known that the expansions of $(\mathbf{Q}_1, P_1)$ for small $\tilde{r}$ are of the form[35]

$$\begin{aligned} \mathbf{Q}_1(\tilde{\mathbf{r}}) &= -[\mathbf{s}(\tilde{\mathbf{r}}) - \mathbf{H}] \cdot \bar{\mathbf{F}}_0 + O(\tilde{r}) \\ P_1(\tilde{\mathbf{r}}) &= -\mathbf{t}(\tilde{\mathbf{r}}) \cdot \bar{\mathbf{F}}_0 + O(\tilde{r}^{-1}) \end{aligned} \tag{4.20}$$

where $\mathbf{H}$ is a constant second-rank tensor (Appendix C).

4.5. First-Order Inner Approximation

The boundary conditions of $(\mathbf{q}_1, p_1)$ as $r \to \infty$ are obtained from (4.20) as

$$\mathbf{q}_1 \sim \mathbf{H} \cdot \bar{\mathbf{F}}_0, \qquad p_1 = o(r^{-1})$$

The asymptotic forms of the solution for the Stokes problem (4.8), (4.9), and (4.21) are easily found to be

$$\begin{aligned} \mathbf{q}_1(\mathbf{r}) &= \mathbf{H} \cdot \bar{\mathbf{F}}_0 - \mathbf{s}(\mathbf{r}) \cdot \mathbf{b} + O(r^{-2}) \\ p_1(\mathbf{r}) &= -\mathbf{t}(\mathbf{r}) \cdot \mathbf{b} + O(r^{-3}) \end{aligned} \tag{4.22}$$

where $\mathbf{b}$ is a constant vector.

4.6. Systematic Force and Torque

It has already been seen that the zeroth-order inner approximation (4.13) and the the first order (4.22) are both solutions of the Stokes equations. As is well known, in low-Reynolds-number hydrodynamics,[32] the force and torque acting on a particle of an arbitrary shape are given by

$$\begin{aligned} \bar{\mathbf{F}}_0 &= -\mathbf{\Gamma} \cdot \mathbf{U}_B - \mathbf{\Lambda} \cdot (\mathbf{\Omega}_B - \mathbf{\Omega}_f) \\ \bar{\mathbf{M}}_0 &= -\mathbf{\Lambda}^t \cdot \mathbf{U}_B - \mathbf{\Sigma} \cdot (\mathbf{\Omega}_B - \mathbf{\Omega}_f) \end{aligned} \tag{4.23}$$

and

$$\bar{\mathbf{F}}_1 = \mathbf{\Gamma} \cdot \mathbf{H} \cdot \bar{\mathbf{F}}_0, \qquad \bar{\mathbf{M}}_1 = \mathbf{\Lambda}^t \cdot \mathbf{H} \cdot \bar{\mathbf{F}}_0 \tag{4.24}$$

The tensor $\mathbf{\Gamma}$ is a translation dyadic which depends only upon the shape of the particle. The tensors $\mathbf{\Sigma}$ and $\mathbf{\Lambda}$ are the rotation dyadic and the coupling dyadic at the origin, which depend on the particle shape and the location of the origin O. The tensors $\mathbf{\Gamma}$ and $\mathbf{\Sigma}$ have the following symmetry relations:

$$\Gamma_{ij} = \Gamma_{ji}, \qquad \Sigma_{ij} = \Sigma_{ji} \tag{4.25}$$

5. THE ADJOINT FIELD

5.1. Inner and Outer Expansions

We write $(q^*_{im}, p^*_m) = (\mathbf{q}^*, \mathbf{p}^*)$ for the sake of simplicity and analyze the adjoint field in the same way as in Section 4. Taking into account the expansions of the systematic field $(\bar{\mathbf{v}}, \bar{p})$ for small $R_\kappa^{1/2}$, we write the inner expansions as follows:

$$\begin{aligned} \mathbf{q}^* &= \mathbf{q}_0^*(\mathbf{r}) + R_\kappa^{1/2}\mathbf{q}_1^*(\mathbf{r}) + o(R_\kappa^{1/2}) \\ \mathbf{p}^* &= \mathbf{p}_0^*(\mathbf{r}) + R_\kappa^{1/2}\mathbf{p}_1^*(\mathbf{r}) + o(R_\kappa^{1/2}) \end{aligned} \tag{5.1}$$

We define a tensor f_{im} due to the adjoint field (q_{im}^*, p_m^*) by

$$\mathbf{f} \equiv f_{im} = \int_{S_p} \theta_{ilm}^* \, dS_l \tag{5.2}$$

in which f_{im} is the ith component of the dimensionless Stokes force acting on a particle translated along the mth axis with unit velocity. The expansion of (5.2) in $R_\kappa^{1/2}$ is

$$\mathbf{f} = \mathbf{f}_0 + R_\kappa^{1/2}\mathbf{f}_1 + o(R_\kappa^{1/2}) \tag{5.3}$$

where $\mathbf{f}_0$ and $\mathbf{f}_1$ are tensors due to $(\mathbf{q}_0^*, \mathbf{p}_0^*)$ and $(\mathbf{q}_1^*, \mathbf{p}_1^*)$, respectively. The outer expansions are

$$\begin{aligned} \mathbf{q}^* &= R_\kappa^{1/2}\mathbf{Q}_1^*(\tilde{\mathbf{r}}) + o(R_\kappa^{1/2}) \\ \mathbf{p}^* &= R_\kappa \mathbf{P}_1^*(\tilde{\mathbf{r}}) + o(R_\kappa) \end{aligned} \tag{5.4}$$

Substituting (4.1), (4.4), and (5.1) into (3.9) and (3.10), we obtain

$$\Delta \mathbf{q}_0^* - \nabla \mathbf{p}_0^* = 0, \qquad \nabla \cdot \mathbf{q}_0^* = 0 \tag{5.5}$$

$$\begin{aligned} \mathbf{q}_0^* &= \mathbf{I}, \qquad && \mathbf{r} \text{ on } S_p \\ &\to 0 && \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{5.6}$$

to the order of R_κ^0 and

$$\Delta \mathbf{q}_1^* - \nabla \mathbf{p}_1^* = 0, \qquad \nabla \cdot \mathbf{q}_1^* = 0 \tag{5.7}$$

$$\mathbf{q}_1^* = 0, \qquad \mathbf{r} \text{ on } S_p \tag{5.8}$$

to the order of $R_\kappa^{1/2}$. The outer boundary conditions for $\mathbf{q}_1^*$ are established by the matching conditions. For $(\mathbf{Q}_1^*, \mathbf{P}_1^*)$ we have the equations

$$-(\mathbf{C} \cdot \tilde{\mathbf{r}}) \cdot \tilde{\nabla} \mathbf{Q}_1^* + \mathbf{C}' \cdot \mathbf{Q}_1^* = -\tilde{\nabla} \mathbf{P}_1^* + \tilde{\Delta} \mathbf{Q}_1^*, \qquad \tilde{\nabla} \cdot \mathbf{Q}_1^* = 0 \tag{5.9}$$

$$\mathbf{Q}_1^*(\tilde{\mathbf{r}}) \to 0 \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to \infty \tag{5.10}$$

with the matching conditions at $\tilde{r} = 0$.

5.2. Zeroth-Order Inner Approximation

The solution of (5.5) and (5.6) can be expressed in terms of the Stokes solution. The asymptotic forms of the solution as $r \to \infty$ are

$$\begin{aligned} \mathbf{q}_0^* &= -\mathbf{s}(\mathbf{r}) \cdot \mathbf{f}_0 - [\nabla \mathbf{s}(\mathbf{r})] : \mathbf{B}_0^* + O(r^{-3}) \\ \mathbf{p}_0^*(\mathbf{r}) &= -\mathbf{t}(\mathbf{r}) \cdot \mathbf{f}_0 - [\nabla \mathbf{t}(\mathbf{r})] : \mathbf{B}_0^* + O(r^{-4}) \end{aligned} \tag{5.11}$$

with

$$(\mathbf{B}_0^*)_{ij} - (\mathbf{B}_0^*)_{ji} = \varepsilon_{ijl} (\mathbf{f}_0)_l \tag{5.12}$$

It follows from (4.23) that, recalling the coupling between the force and torque,

$$(f_0)_{ij} = -\Gamma_{ij}, \qquad (\lambda_0)_{ij} = -\Lambda_{ij}, \qquad (m_0)_{ij} = -\Sigma_{ij} \tag{5.13}$$

where λ_0 and m_0 are the Stokes coupling and torque tensor defined as in (5.2).

5.3. First-Order Outer Approximation

From the asymptotic forms (5.11) the matching conditions of $(\mathbf{Q}_1^*, \mathbf{P}_1^*)$ at $\tilde{r} = 0$ are obtained as

$$\mathbf{Q}_1^*(\tilde{\mathbf{r}}) = -\mathbf{s}(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0, \qquad \mathbf{P}_1^*(\tilde{\mathbf{r}}) = -\mathbf{t}(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0 \tag{5.14}$$

The solution of (5.9) and (5.10) satisfying (5.14) can be written as

$$\mathbf{Q}_1^*(\tilde{\mathbf{r}}) = -\mathbf{G}^*(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0, \qquad \mathbf{P}_1^*(\tilde{\mathbf{r}}) = -\mathbf{T}^*(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0 \tag{5.15}$$

where $(\mathbf{G}^*, \mathbf{T}^*)$ is the solution of

$$-(\mathbf{C} \cdot \tilde{\mathbf{r}}) \cdot \tilde{\nabla} \mathbf{G}^* + \mathbf{C}' \cdot \mathbf{G}^* = -\tilde{\nabla} \mathbf{T}^* + \tilde{\Delta} \mathbf{G}^* + \mathbf{I} \delta(\tilde{\mathbf{r}}), \qquad \tilde{\nabla} \cdot \mathbf{G}^* = 0 \tag{5.16}$$

$$\mathbf{G}^*(\tilde{\mathbf{r}}) \to 0 \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to \infty \tag{5.17}$$

The expansions for small $\tilde{r}$ yield

$$\begin{aligned} \mathbf{Q}_1^*(\tilde{\mathbf{r}}) &= -[\mathbf{s}(\tilde{\mathbf{r}}) - \mathbf{H}^*] \cdot \mathbf{f}_0 + O(\tilde{r}) \\ \mathbf{P}_1^*(\tilde{\mathbf{r}}) &= -\mathbf{t}(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0 + O(\tilde{r}^{-1}) \end{aligned} \tag{5.18}$$

where $\mathbf{H}^*$ is a constant second-rank tensor (Appendix D).

5.4. First-Order Inner Approximation

From the expansions (5.18) and the matching principle, the first-order inner solution should satisfy the boundary condition

$$\mathbf{q}_1^* \to \mathbf{H}^* \cdot \mathbf{f}_0 \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to \infty \tag{5.19}$$

The asymptotic forms of the Stokes solution of (5.7), (5.8), and (5.19) for large r are

$$\begin{aligned} \mathbf{q}_1^*(\mathbf{r}) &= \mathbf{H}^* \cdot \mathbf{f}_0 - \mathbf{s}(\mathbf{r}) \cdot \mathbf{b}^* + O(r^{-2}) \\ \mathbf{p}_1^*(\mathbf{r}) &= -\mathbf{t}(\mathbf{r}) \cdot \mathbf{b}^* + O(r^{-3}) \end{aligned} \tag{5.20}$$

where $\mathbf{b}^*$ is a constant tensor.

6. The FIELD $(\langle \tilde{v}_i(\mathbf{x}) \tilde{F}_l \rangle, \langle \tilde{p}(\mathbf{x}) \tilde{F}_l \rangle)$

6.1. Inner and Outer Expansions

Let us write

$$\langle \tilde{v}_i(\mathbf{x}) \tilde{F}_l \rangle \equiv w_{il}(\mathbf{r}) = \mathbf{w}(\mathbf{r}), \qquad \langle \tilde{p}(\mathbf{x}) \tilde{F}_l \rangle \equiv \pi_l(r) = \boldsymbol{\pi}(\mathbf{r}) \tag{6.1}$$

The analysis of (3.5) with (3.6) will proceed in the same way as those of Sections 4 and 5. The inner expansions are

$$\begin{aligned} \mathbf{w}(\mathbf{r}) &= \mathbf{w}_0(\mathbf{r}) + R_\kappa^{1/2} \mathbf{w}_1(\mathbf{r}) + o(R_\kappa^{1/2}) \\ \boldsymbol{\pi}(\mathbf{r}) &= \boldsymbol{\pi}_0(\mathbf{r}) + R_\kappa^{1/2} \boldsymbol{\pi}_1(\mathbf{r}) + o(R_\kappa^{1/2}) \end{aligned} \tag{6.2}$$

and the outer expansions

$$\begin{aligned} \mathbf{w}(\tilde{\mathbf{r}}) &= R_\kappa^{1/2} \mathbf{W}_1(\tilde{\mathbf{r}}) + o(R_\kappa^{1/2}) \\ \boldsymbol{\pi}(\tilde{\mathbf{r}}) &= R_\kappa \boldsymbol{\Pi}_1(\tilde{\mathbf{r}}) + o(R_\kappa) \end{aligned} \tag{6.3}$$

Using (3.18) and substituting (4.1), (4.4), and (5.1) into (3.5) and (3.6), we obtain

$$\Delta \mathbf{w}_0 - \nabla \boldsymbol{\pi}_0 = \Delta \mathbf{q}_0^*, \qquad \nabla \cdot \mathbf{w}_0 = 0 \tag{6.4}$$

$$\begin{aligned} \mathbf{w}_0 &= 0, \qquad && \mathbf{r} \text{ on } S_p \\ &\to 0 && \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{6.5}$$

to the lowest order in $R_\kappa^{1/2}$ and

$$\Delta \mathbf{w}_1 - \nabla \pi_1 = \Delta \mathbf{q}_1^*, \qquad \nabla \cdot \mathbf{w}_1 = 0 \tag{6.6}$$

$$\mathbf{w}_1 = 0, \qquad \mathbf{r} \text{ on } S_p \tag{6.7}$$

to the first order. The outer boundary conditions for $(\mathbf{w}_1, \pi_1)$ can also be derived by the matching conditions. The equations and boundary conditions for $(\mathbf{W}_1, \mathbf{\Pi}_1)$ are

$$(\mathbf{C} \cdot \tilde{\mathbf{r}}) \cdot \tilde{\nabla} \mathbf{W}_1 + \mathbf{C} \cdot \mathbf{W}_1 = -\tilde{\nabla} \mathbf{\Pi}_1 + \tilde{\Delta} \mathbf{W}_1 - \tilde{\Delta} \mathbf{Q}_1^*, \qquad \tilde{\nabla} \cdot \mathbf{W}_1 = 0 \tag{6.8}$$

$$\mathbf{W}_1(\tilde{\mathbf{r}}) \to 0 \qquad \text{as} \quad |\tilde{\mathbf{r}}| \to \infty \tag{6.9}$$

and the matching conditions at $\tilde{r} = 0$.

6.2. Zeroth-Order Inner Approximation

The solution of (6.4) and (6.5) is given by

$$\begin{aligned} (w_0)_{il}(\mathbf{r}) &= \langle \tilde{v}_i(\mathbf{r}) \tilde{F}_l \rangle_0 = 0 \\ (\pi_0)_l(\mathbf{r}) &= \langle \tilde{p}(\mathbf{r}) \tilde{F}_l \rangle_0 = -(p_0^*)_l(\mathbf{r}) \\ &= t_j(\mathbf{r})(f_0)_{jl} + O(r^{-3}) \end{aligned} \tag{6.10}$$

6.3. First-Order Outer Approximation

The matching conditions for $(\mathbf{W}(\tilde{\mathbf{r}}), \mathbf{\Pi}(\tilde{\mathbf{r}}))$ at $\tilde{r} = 0$ are

$$\mathbf{W}_1(\tilde{\mathbf{r}}) = o(\tilde{r}^{-1}), \qquad \mathbf{\Pi}_1(\tilde{\mathbf{r}}) = \mathbf{t}(\tilde{\mathbf{r}}) \cdot \mathbf{f}_0 \tag{6.11}$$

The solution of (6.8) and (6.9) which satisfies (6.11) is given by

$$\begin{aligned} \mathbf{W}_1(\tilde{\mathbf{r}}) &= -\tfrac{1}{2}[\mathbf{G}^*(\tilde{\mathbf{r}}) - \mathbf{G}(\tilde{\mathbf{r}})] \cdot \mathbf{f}_0 \\ \mathbf{\Pi}_1(\tilde{\mathbf{r}}) &= -\mathbf{P}_1^*(\tilde{\mathbf{r}}) - \tfrac{1}{2}[\mathbf{T}^*(\tilde{\mathbf{r}}) - \mathbf{T}(\tilde{\mathbf{r}})] \cdot \mathbf{f}_0 \end{aligned} \tag{6.12}$$

(Appendix E). From (4.20) and (5.18) the expansion of $\mathbf{W}_1(\tilde{\mathbf{r}})$ at $\tilde{r} = 0$ is

$$\mathbf{W}_1(\tilde{\mathbf{r}}) = \tfrac{1}{2}(\mathbf{H}^* - \mathbf{H}) \cdot \mathbf{f}_0 + O(\tilde{r}) \tag{6.13}$$

6.4. First-Order Inner Approximation

By the matching principle, the field $(\mathbf{w}_1(\mathbf{r}), \pi_1(\mathbf{r}))$ must satisfy the condition

$$\mathbf{w}_1(\mathbf{r}) \to \tfrac{1}{2}(\mathbf{H}^* - \mathbf{H}) \cdot \mathbf{f}_0 \qquad \text{as} \quad |\mathbf{r}| \to \infty \tag{6.14}$$

Then it is easily found that the asymptotic expressions of the solution for the Stokes problem (6.6), (6.7), and (6.14) are

$$\begin{aligned}(w_1)_{il}(\mathbf{r}) &= \langle \tilde{v}_i(\mathbf{r})\tilde{F}_l\rangle_1 = \tfrac{1}{2}(H^*_{ij} - H_{ij})(f_0)_{jl} - s_{ij}(\mathbf{r})d_{jl} + O(r^{-2}) \\ (\pi_1)_l(\mathbf{r}) &= \langle \tilde{p}(\mathbf{r})\tilde{F}_l\rangle_1 = t_j(\mathbf{r})d_{jl} + O(r^{-3}) \qquad \text{as} \quad |\mathbf{r}| \to \infty\end{aligned} \tag{6.15}$$

where d_{jl} is a constant tensor.

7. THE CORRELATION FUNCTION OF THE FLUCTUATING FORCES

Let us consider the correlation function of the fluctuating forces. Corresponding to the expansions of (w_{il}, π_l), the correlation function is expanded as

$$Y_{ij} = (Y_0)_{ij} + R_\kappa^{1/2}(Y_1)_{ij} + o(R_\kappa^{1/2}) \tag{7.1}$$

where

$$(Y_0)_{ij} = \int_{S_p} [\langle \tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0 + \langle \tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0]\, dS_k(\mathbf{r}) \tag{7.2}$$

is due to $(\mathbf{w}_0, \pi_0)$ and the second term of (7.1) due to $(\mathbf{w}_1, \pi_1)$ is also defined similarly. Noting (3.18), $\nabla \cdot \mathbf{q}_0^* = 0$, and (6.1), we can write the equation of motion (6.4) as

$$\Delta\langle \tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_0 - \frac{\partial}{\partial x_i}\langle \tilde{p}(\mathbf{r})\tilde{F}_j\rangle_0 = -\frac{\partial}{\partial r_k}\langle \tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0 \tag{7.3}$$

where

$$\langle \tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0 = -\left[\frac{\partial}{\partial r_k}(q_0^*)_{ij}(\mathbf{r}) + \frac{\partial}{\partial r_i}(q_0^*)_{kj}(\mathbf{r})\right] \tag{7.4}$$

With the use of the stress tensor form (2.10), one can write Eq. (7.3) as

$$\begin{aligned}0 &= \frac{\partial}{\partial r_k}[\langle \tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0 + \langle \tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0] \\ &= \frac{\partial}{\partial r_k}\left\{-\langle \tilde{p}(\mathbf{r})\tilde{F}_j\rangle_0\delta_{ik} + \frac{\partial}{\partial r_k}[\langle \tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_0 - (q_0^*)_{ij}(\mathbf{r})]\right. \\ &\quad \left. + \frac{\partial}{\partial r_i}[\langle \tilde{v}_k(\mathbf{r})\tilde{F}_j\rangle_0 - (q_0^*)_{kj}(\mathbf{r})]\right\}\end{aligned} \tag{7.5}$$

Substituting (6.10) into (7.5) and using (3.11), we obtain

$$0=\frac{\partial}{\partial r_k}\left[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_0\right]=-\frac{\partial}{\partial r_k}(\theta_0^*)_{ikj}(\mathbf{r}) \tag{7.6}$$

Similarly, for $(\mathbf{w}_1, \pi_1)$ we have

$$\begin{aligned}0&=\frac{\partial}{\partial r_k}\left[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1\right]\\&=\frac{\partial}{\partial r_k}\left\{-\langle\tilde{p}(\mathbf{r})\tilde{F}_j\rangle_1\delta_{ik}+\frac{\partial}{\partial r_k}\left[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})\right]\right.\\&\qquad\left.+\frac{\partial}{\partial r_i}\left[\langle\tilde{v}_k(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{kj}(\mathbf{r})\right]\right\}\end{aligned} \tag{7.7}$$

7.1. Zeroth-Order Correlation Function

Equation (7.6) implies that the integrand of (7.2) is given in terms of the stress tensor due to the zeroth-order adjoint field. Thus, from (5.2) we obtain

$$(Y_0)_{ij}=-\int_{S_p}(\theta_0^*)_{ikj}(\mathbf{r})\,dS_k(\mathbf{r})=-(f_0)_{ij} \tag{7.8}$$

7.2. First-Order Correlation Function

The first-order correlation function can be obtained as follows.[34] Consider the identities

$$\begin{aligned}0=(q_0^*)_{im}(\mathbf{r})\frac{\partial}{\partial r_k}&\left[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1\right]\\&-\left[\frac{\partial}{\partial r_k}(\theta_0^*)_{ikm}(\mathbf{r})\right]\left[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})\right]\end{aligned} \tag{7.9}$$

which follows from (7.7) and the fact that Eq. (5.5) can be written as

$$\frac{\partial}{\partial r_k}(\theta_0^*)_{ikm}(\mathbf{r})=0 \tag{7.10}$$

Noting the solenoidal condition in i of $(q_0^*)_{im}(\mathbf{r})$, $\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1$, and $(q_1^*)_{ij}(\mathbf{r})$ and using a similar relation to (3.13),

$$\begin{aligned}&\left[\frac{\partial}{\partial r_k}(q_0^*)_{im}(\mathbf{r})\right]\left[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1\right]\\&\qquad=(\theta_0^*)_{ikm}(\mathbf{r})\frac{\partial}{\partial r_k}\left[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})\right]\end{aligned} \tag{7.11}$$

we can write (7.9) as

$$0=\frac{\partial}{\partial r_k}\{(q_0^*)_{im}(\mathbf{r})[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1] - (\theta_0^*)_{ikm}(\mathbf{r})[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})]\} \tag{7.12}$$

Integrating this over the volume V_A bounded externally by the surface S_A of radius A and by the surface S_p of the particle yields

$$\left(\int_{S_A}-\int_{S_p}\right)\{(q_0^*)_{im}(\mathbf{r})[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1] - (\theta_0^*)_{ikm}(\mathbf{r})[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})]\}\,dS_k(\mathbf{r})=0 \tag{7.13}$$

Using the boundary condition (5.6), (5.8), and (6.7), and letting $A\to\infty$, gives

$$\begin{aligned}(Y_1)_{mj}&=\int_{S_p}[\langle\tilde{\tau}_{mk}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{mk}(\mathbf{r})\tilde{F}_j\rangle_1]\,dS_k(\mathbf{r})\\ &=(I_1)_{mj}+(I_2)_{mj}\end{aligned} \tag{7.14}$$

where

$$(I_1)_{mj}=\lim_{A\to\infty}\int_{S_A}(q_0^*)_{im}(\mathbf{r})[\langle\tilde{\tau}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1+\langle\tilde{\sigma}_{ik}(\mathbf{r})\tilde{F}_j\rangle_1]\,dS_k(\mathbf{r}) \tag{7.15}$$

$$(I_2)_{mj}=-\lim_{A\to\infty}\int_{S_A}(\theta_0^*)_{ikm}(\mathbf{r})[\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})]\,dS_k(\mathbf{r}) \tag{7.16}$$

Note that the terms in the integrand only contribute to $(I_1)_{mj}$ if they do not tend to zero faster than r^{-2} as $r\to\infty$. From the asymptotic forms (5.11), (5.20), and (6.15) for large r, it follows that the integrand of (7.15) is $O(r^{-3})$. Thus we obtain

$$(I_1)_{mj}=0 \tag{7.17}$$

Since $\langle\tilde{v}_i(\mathbf{r})\tilde{F}_j\rangle_1-(q_1^*)_{ij}(\mathbf{r})=O(1)$ for large r from (5.20) and (6.15), the contributions to the integral (7.16) come only from the term $(q_0^*)_{im}(\mathbf{r})=O(r^{-1})$ as $r\to\infty$. Therefore substitution of (5.20) and (6.15) into (7.16) gives

$$\begin{aligned}(I_2)_{mj}&=-[\tfrac{1}{2}(H_{ik}^*-H_{ik})(f_0)_{kj}-H_{ik}^*(f_0)_{kj}]\lim_{A\to\infty}\int_{S_A}(\theta_0^*)_{ilm}(\mathbf{r})\,dS_l(\mathbf{r})\\ &=\tfrac{1}{2}(H_{ik}^*+H_{ik})(f_0)_{kj}\lim_{A\to\infty}\int_{S_A}(\theta_0^*)_{ilm}(\mathbf{r})\,dS_l(\mathbf{r})\end{aligned} \tag{7.18}$$

On the other hand, integrating (7.10) over the volume V_A and letting $A \to \infty$, we obtain

$$\int_{S_p} (\theta_0^*)_{ilm}(\mathbf{r})\, dS_l(\mathbf{r}) = (f_0)_{im} = \lim_{A\to\infty} \int_{S_A} (\theta_0^*)_{ilm}(\mathbf{r})\, dS_l(\mathbf{r}) \tag{7.19}$$

and thus

$$(I_2)_{mj} = \tfrac{1}{2}(H_{ik}^* + H_{ik})(f_0)_{im}(f_0)_{kj} \tag{7.20}$$

Substituting (7.17) and (7.20) into Eq. (7.14), we finally obtain

$$(Y_1)_{mj} = \tfrac{1}{2}(H_{ik}^* + H_{ik})(f_0)_{im}(f_0)_{kj} \tag{7.21}$$

The correlation functions for the fluctuating torques or force-torque can be derived similarly.

8. THE FD THEOREM FOR A BROWNIAN PARTICLE IN A ROTATING FLOW

Substituting (4.23) and (4.24) into (2.18) and changing the coordinate system to the laboratory frame, we obtain the Langevin equations including the effects of particle rotation in matrix form as

$$\frac{d}{dt}\begin{pmatrix} \mathbf{V}_B \\ \mathbf{J}\cdot\mathbf{\Omega}_B \end{pmatrix} = -\begin{pmatrix} \mathbf{I} + R_\kappa^{1/2}\mathbf{\Gamma}\cdot\mathbf{H}, & 0 \\ R_\kappa^{1/2}\mathbf{\Lambda}^t\cdot\mathbf{H}, & \mathbf{I} \end{pmatrix}\begin{pmatrix} \mathbf{\Gamma}, & \mathbf{\Lambda} \\ \mathbf{\Lambda}^t, & \mathbf{\Sigma} \end{pmatrix}\begin{pmatrix} \mathbf{V}_B - \mathbf{V}_f \\ \mathbf{\Omega}_B - \mathbf{\Omega}_f \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{F}} \\ \tilde{\mathbf{M}} \end{pmatrix} \tag{8.1}$$

up to $O(R_\kappa^{1/2})$, where $\mathbf{V}_B = (d/dt)\mathbf{X}_B$, $\mathbf{V}_f = \kappa\mathbf{C}\cdot\mathbf{X}_B$, and $\mathbf{X}_B$ is the position vector of the particle from the origin O which is on the axis of fluid rotation. Substitution of (7.8) and (7.21) into (7.1) and the use of (5.13) yields

$$\langle\tilde{\mathbf{F}}\rangle = o(R_\kappa^{1/2}), \qquad \langle\tilde{\mathbf{M}}\rangle = o(R_\kappa^{1/2}) \tag{8.2}$$

$$\int_{-\infty}^{\infty} \langle\tilde{F}_i(t)\,\tilde{F}_j(0)\rangle\, dt = \Gamma_{ij} + \tfrac{1}{2}R_\kappa^{1/2}\Gamma_{ik}(H_{kl} + H_{kl}^*)\Gamma_{lj} + o(R_\kappa^{1/2}) \tag{8.3}$$

$$\int_{-\infty}^{\infty} \langle\tilde{F}_i(t)\,\tilde{M}_j(0)\rangle\, dt = \Lambda_{ij} + \tfrac{1}{2}R_\kappa^{1/2}\Gamma_{ik}(H_{kl} + H_{kl}^*)\Lambda_{lj} + o(R_\kappa^{1/2}) \tag{8.4}$$

$$\int_{-\infty}^{\infty} \langle\tilde{M}_i(t)\,\tilde{M}_j(0)\rangle\, dt = \Sigma_{ij} + \tfrac{1}{2}R_\kappa^{1/2}\Lambda_{ki}(H_{kl} + H_{kl}^*)\Lambda_{lj} + o(R_\kappa^{1/2}) \tag{8.5}$$

where

$$H_{ij} = H_{ji}^* = \begin{pmatrix} h_1 & 0 & -h_3 \\ 0 & h_2 & 0 \\ h_3 & 0 & h_1 \end{pmatrix} \tag{8.6}$$

$$6\pi h_1 = 3\,\frac{\sqrt{2}(19+9\sqrt{3})}{280}, \qquad 6\pi h_2 = \frac{4}{7}, \qquad 6\pi h_3 = 3\,\frac{\sqrt{2}(19-9\sqrt{3})}{280} \tag{8.7}$$

Thus, the FD theorems hold up to $R_\kappa^{1/2}$ for a Brownian particle of arbitrary shape in a rotating fluid. It should be noted that for a particle with $\mathbf{\Lambda} = 0$ (e.g., sphere, regular polyhedron, ellipsoid), inertial effects due to fluid rotation do not affect the systematic and fluctuating torque to $R_\kappa^{1/2}$. Formulas (8.1)–(8.7) can be applied to any Brownian particle of arbitrary shape, but it is not easy to obtain the explicit form of the friction tensors for a particle. For the special case of a spherical particle in which $\Gamma_{ij} = 6\pi\delta_{ij}$, $\Lambda_{ij} = 0$, and $\Sigma_{ij} = 8\pi\delta_{ij}$, the Langevin equations can be written in dimensional form as

$$m\frac{d}{dt'}\mathbf{V}'_B = -6\pi\mu a(\mathbf{I} + 6\pi R_\kappa^{1/2}\mathbf{H})\cdot(\mathbf{V}'_B - \mathbf{C}'\cdot\mathbf{X}'_B) + \tilde{\mathbf{F}}' \tag{8.8}$$

$$ma^2\mathbf{J}\cdot\frac{d}{dt'}\mathbf{\Omega}'_B = -8\pi\mu a^3(\mathbf{\Omega}'_B - \mathbf{\Omega}'_f) + \tilde{\mathbf{M}}' \tag{8.9}$$

and the FD theorems are

$$\langle\tilde{\mathbf{F}}'\rangle = \mu a U_0 o(R_\kappa^{1/2}), \qquad \langle\tilde{\mathbf{M}}'\rangle = \mu a^3 \Omega_0 o(R_\kappa^{1/2}) \tag{8.10}$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i'(t)\,\tilde{F}_j'(0)\rangle\,dt = 2k_B T 6\pi\mu a[\delta_{ij} + \tfrac{1}{2}R_\kappa^{1/2} 6\pi(H_{ij} + H_{ij}^*) + o(R_\kappa^{1/2})] \tag{8.11}$$

$$\int_{-\infty}^{\infty} \langle \tilde{F}_i'(t)\,\tilde{M}_j'(0)\rangle\,dt = k_B T\mu a^2 o(R_\kappa^{1/2}) \tag{8.12}$$

$$\int_{-\infty}^{\infty} \langle \tilde{M}_i'(t)\,\tilde{M}_j'(0)\rangle\,dt = 2k_B T 8\pi\mu a^3[\delta_{ij} + o(R_\kappa^{1/2})] \tag{8.13}$$

9. THE FD THEOREM FOR A BROWNIAN PARTICLE IN A CENTRIFUGE

Recently Ryskin[30] studied diffusion in a rapidly rotating flow on a phenomenological basis and found that the diffusion coefficients depend on the angular velocity of the fluid and become anisotropic. We shall show briefly that his results can also be derived from the semimicroscopic level of the coarse graining by using the FHD. As in previous sections, consider a particle in a rotating fluid. We choose the coordinate system to translate with the velocity $\mathbf{U}'_B$ and to rotate with the angular velocity $\mathbf{\Omega}'_f$ along the x'_2 direction, and use the same normalilzation as in Section 2. We assume that the fluid rotation is so fast that the time derivative $\partial \mathbf{v}'/\partial t'$ can be neglected as compared to the Coriolis force $2\mathbf{\Omega}'_f \times \mathbf{v}'$, which implies $U_0/\Omega'_f a \ll 1$ (centrifuge condition). The convective terms are also assumed to be small as compared to the Coriolis term. The condition

$$U_0/\Omega'_f a \ll T_a^{1/2} \ll 1 \tag{9.1}$$

is different from the condition for the case of the slow rotation (2.22), $R \ll R_\kappa^{1/2} \ll 1$, where $T_a = 2\Omega'_f a^2/\nu$ is the Taylor number. Under the condition (9.1), the equations and the boundary conditions corresponding to (2.14) and (2.15) are given by

$$T_a \mathbf{e}_2 \times \bar{\mathbf{v}} = -\nabla \bar{p} + \Delta \bar{\mathbf{v}}, \qquad \nabla \cdot \bar{\mathbf{v}} = 0 \tag{9.2}$$

$$\begin{aligned} \bar{\mathbf{v}} &= (\mathbf{\Omega}_B - \mathbf{\Omega}_f) \times \mathbf{r}, && \mathbf{r} \text{ on } S_p \\ &\to -\mathbf{U}_B && \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{9.3}$$

respectively, where $\mathbf{e}_2$ is the unit vector in the x_2 direction. Similarly, the equations corresponding to (3.3) and (3.5) are those in which $R(\delta_{ij}\bar{\mathbf{v}} \cdot \nabla + \partial \bar{v}_i/\partial x_j)$ is replaced by $T_a \varepsilon_{i2j}$, and the equation corresponding to (3.9) becomes one in which $-R(\delta_{ij}\bar{\mathbf{v}} \cdot \nabla - \partial \bar{v}_j/\partial x_i)$ is replaced by $-T_a \varepsilon_{i2j}$. The analysis in terms of the expansions in $T_a^{1/2}$ can proceed similarly as in Sections 4–6 (see ref. 38), and yields the same results as those in Sections 4–6 if the Reynolds number $R_\kappa^{1/2}$ and the tensor $H_{ij} = H_{ji}^*$ are replaced by $T_a^{1/2}$ and by

$$6\pi h_1 = \frac{1}{\sqrt{2}} \frac{5}{7}, \qquad 6\pi h_2 = \frac{1}{\sqrt{2}} \frac{4}{7}, \qquad 6\pi h_3 = \frac{1}{\sqrt{2}} \frac{3}{5} \tag{9.4}$$

respectively. Thus, the Langevin equations for a particle of arbitrary shape are given by

$$\frac{d}{dt}\begin{pmatrix} \mathbf{V}_B \\ \mathbf{J} \cdot \mathbf{\Omega}_B \end{pmatrix} = -\begin{pmatrix} \mathbf{I} + T_a^{1/2} \mathbf{\Gamma} \cdot \mathbf{H}, & 0 \\ T_a^{1/2} \mathbf{\Lambda}^t \cdot \mathbf{H}, & \mathbf{I} \end{pmatrix} \begin{pmatrix} \mathbf{\Gamma}, & \mathbf{\Lambda} \\ \mathbf{\Lambda}^t, & \mathbf{\Sigma} \end{pmatrix} \begin{pmatrix} \mathbf{V}_B \\ \mathbf{\Omega}_B - \mathbf{\Omega}_f \end{pmatrix} + \begin{pmatrix} \tilde{\mathbf{F}} \\ \tilde{\mathbf{M}} \end{pmatrix} \tag{9.5}$$

up to $O(T_a^{1/2})$, where $\mathbf{V}_B = (d/dt)\mathbf{X}_B$ and $\mathbf{X}_B$ is the position vector from the origin, and the FD theorems are again given by (8.2)–(8.6) and (9.4) with replacement of $R_\kappa^{1/2}$ by $T_a^{1/2}$.

10. DISCUSSION

We have derived the Langevin equations for a Brownian particle of arbitrary shape in an unbounded rotating fluid by a semimicroscopic approach based on the FHD. In order to obtain the approximate solution valid over the entire flow region, we used matched asymptotic expansions in small Reynolds numbers ($R \ll R_\kappa^{1/2} \ll 1$) or in small Taylor numbers ($U_0/\Omega'_f a \ll T_a^{1/2} \ll 1$), and obtained the systematic force and torque and the FD theorems valid up to $R_\kappa^{1/2}(T_a^{1/2})$. The Langevin equations may be interpreted as follows. For simplicity, consider a spherical particle [(8.8)–(8.13)]. Constant tensors $\mathbf{H}$ and $\mathbf{H}^*$ can be seen as a uniform flow in the ith direction that a fluid sphere of radius $r_* = R_\kappa^{-1/2}(T_a^{-1/2}) \gg 1$ experiences when the particle moves along the jth direction with unit velocity. The particle dissipates its kinetic energy due to friction on its surface and also on the fluid surface at r_*. The random force of zeroth order acting on the particle is due to the thermal agitation within this fluid sphere and balances the zeroth-order particle drag. A fluid sphere of radius r_* moves with a velocity of the order of $R_\kappa^{1/2}(T_a^{1/2})$ when it is acted on by the random forces at its surface due to thermal agitation in the fluid surrounding the sphere. This agitation balances the drag of the fluid sphere. The latter effects are due to the nonlinear terms of the Navier–Stokes equation. The particle moves randomly within the fluid sphere of radius r_* that undergoes Brownian motion due to the thermal agitation in the rotating fluid. Thus, the FD theorems hold up to the first order of $R_\kappa^{1/2}(T_a^{1/2})$. The above picture suggests that the strength of the nonlinearity is not uniform is space or time, which would also be expected in the coarse-graining processes in other contexts. It should be noted that even if the particle is a sphere, the drag tensor has nondiagonal elements such as h_3, while the correlation functions of the random forces are diagonal, but anisotropic [(8.6), (8.8), and (8.11)]. This is consistent with the fact that the forces perpendicular to the velocity make no contribution to the energy dissipation. The rigid rotation of the whole fluid affects not only the drag and torque of the particle and the correlation functions of the fluctuating forces perpendicular to the angular velocity vector $\mathbf{\Omega}'_f$, but also those parallel to it. For a particle of arbitrary shape the force depends on torque and vice versa because of a nonzero $\mathbf{\Lambda}$ up to $R_\kappa^{1/2}(T_a^{1/2})$, and also for the fluctuating force and torque. If we know the explicit formulas of the Stokes drag and torque on the particle, it is easy to obtain the Langevin equations and the FD theorems valid up to $R_\kappa^{1/2}(T_a^{1/2})$. In the limit of $R_\kappa^{1/2}(T_a^{1/2}) \to 0$, the surface of the fluid sphere at r_* goes to infinity and the additional dissipation at the surface vanishes; thus, our formulas reduce to the well-known equations.[36]

Let us consider the diffusion of a spherical Brownian particle under fluid rotation. The Fokker–Planck equation to the Langevin equations for a spherical Brownian particle (8.8), (8.10), and (8.11) is given by

$$\frac{\partial}{\partial t'} W(\mathbf{X}', t') + \nabla_{\mathbf{x}'}\{\mathbf{C}' \cdot \mathbf{X}' W(\mathbf{X}', t')\}$$
$$= \left[D_\perp \left(\frac{\partial^2}{\partial X_1'^2} + \frac{\partial^2}{\partial X_3'^2} \right) + D_\parallel \frac{\partial^2}{\partial X_2'^2} \right] W(\mathbf{X}', t') \tag{10.1}$$

where $W(\mathbf{X}', t')$ is the probability distribution function of the Brownian particle and

$$D_\perp = \frac{D_0}{1 + 6\pi R_\kappa^{1/2} h_1}, \qquad D_\parallel = \frac{D_0}{1 + 6\pi R_\kappa^{1/2} h_2}, \qquad D_0 = \frac{k_{\mathrm{B}} T}{6\pi\mu a} \tag{10.2}$$

Since $h_2 > h_1 > 0$, the diffusion coefficients $D_\perp$ and $D_\parallel$ get smaller as compared to D_0 due to the fluid rotation, and thus they have weak anisotropy. Solving Eq. (10.1) subject to the conditions

$$W(\mathbf{X}', 0) = \delta(\mathbf{X}'), \qquad W(\mathbf{X}', t') \to 0, \qquad \text{as} \quad |\mathbf{X}'| \to 0 \tag{10.3}$$

we have

$$W(\mathbf{X}', t') = \frac{1}{(4\pi t')^{3/2} (D_\perp^2 D_\parallel)^{1/2}} \exp\left(-\frac{X_1'^2 + X_3'^2}{4 D_\perp t'} - \frac{X_2'^2}{4 D_\parallel t'} \right) \tag{10.4}$$

For the case of rapid rotation the Fokker–Planck equation is given by removing the convective term in (10.1) and replacing the diffusion constants by

$$D_\perp = \frac{D_0}{1 + 6\pi T_a^{1/2} h_1}, \qquad D_\parallel = \frac{D_0}{1 + 6\pi T_a^{1/2} h_2}, \qquad D_0 = \frac{k_{\mathrm{B}} T}{6\pi\mu a} \tag{10.5}$$

These are the same as the results obtained by Ryskin[30] if we take into account the factor 2 in the definition of T_α. It is hoped that the $R_\kappa^{1/2}(\tau 12)$ dependence of the diffusion constants is examined experimentally; e.g., with parameters $k_{\mathrm{B}} = 1.38 \times 10^{-16}$ erg/deg, $T = 300$ K, $\nu = 0.01$ cm^2 sec^{-1} (water), in slow rotation $\Omega_f = 10$ sec^{-1} and $a = 10^{-3}$ cm, and then have $U_0 = 10^{-5/2}$ cm/sec, $R = 10^{-7/2}$, $U_0/\Omega_f' a = 10^{-1/2}$, and $R_\kappa^{1/2} = 10^{-3/2}$; while in rapid rotation $\Omega_f' = 10^3$ sec^{-1} and $a = 5$ μm, and we have $T_a^{1/2} = 0.16$ and $U_0/\Omega_f' a = 10^{-3}$. The correction is small, but detectable.

APPENDIX A. THE EFFECTS OF THE TERMS $R(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}$ ON THE CORRELATION FUNCTIONS

If the $R(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}$ term is retained, terms $R\langle(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle$ and $R\langle(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\tilde{\mathbf{F}}\rangle$ are to be added to (3.3) and (3.5), respectively. Provided that the probability distribution of $\tilde{\boldsymbol{\sigma}}$ is nearly Gaussian, then the triple moment $\langle\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\rangle$ vanishes and the fourth-order moment $\langle\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\rangle$ can be expressed in terms of the second-order moment $\langle\tilde{\boldsymbol{\sigma}}\tilde{\boldsymbol{\sigma}}\rangle$. Similarly, the fourth-order moments $\langle\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\mathbf{F}}\rangle$, $\langle\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle$, etc., can be expressed in terms of the second-order moments as $\langle\tilde{\mathbf{v}}\tilde{\mathbf{v}}\rangle_0$, $\langle\tilde{\mathbf{v}}\tilde{\mathbf{F}}\rangle_0$, $\langle\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle_0$, etc., to the lowest order in $R_\kappa^{1/2}$, which are solutions of (2.16) with all R terms dropped. If we consider the equations for the triple moments such as $\langle\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle$, the R term yields fourth-order moments such as $R\langle(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle$. From the above arguments they can be expressed by the second-order moments as $\langle\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle_0$, which implies that terms like $R<(\tilde{\mathbf{v}}\cdot\nabla)\tilde{\mathbf{v}}\tilde{\boldsymbol{\sigma}}\rangle$ is (3.3) and (3.15) are $O(R^2)$. Similar arguments for the effects on $\langle\tilde{\mathbf{F}}\rangle$ and $\langle\tilde{\mathbf{M}}\rangle$ show that the effect of R terms are $O(R)$.

APPENDIX B. ESTIMATION OF THE SURFACE INTEGRAL IN (3.14)

First we consider the term in (3.14),

$$\mathbf{E}(\mathbf{r}) \equiv -R \lim_{A\to\infty} \int_{S_A} \mathbf{q}^*(\mathbf{x})\cdot\langle\tilde{\mathbf{v}}(\mathbf{x})\,\tilde{\boldsymbol{\sigma}}(\mathbf{r})\rangle\,\bar{\mathbf{v}}(\mathbf{x})\cdot d\mathbf{S}(\mathbf{x}) \tag{B.1}$$

When $\mathbf{x}$ lies at large distance in the outer region, from (4.1), (5.4), and (5.15) we have

$$R\bar{\mathbf{v}}(\mathbf{x}) = R_\kappa^{1/2}\mathbf{C}\cdot\tilde{\mathbf{x}} + \cdots \tag{B.2}$$

$$\mathbf{q}^*(\mathbf{x}) = R_\kappa^{1/2}\mathbf{G}^*(\tilde{\mathbf{x}})\cdot\mathbf{f}_0 + \cdots \tag{B.3}$$

while, if we put

$$\langle\tilde{\mathbf{v}}(\mathbf{x})\,\tilde{\boldsymbol{\sigma}}(\mathbf{r})\rangle \equiv -\nabla_{\mathbf{r}}\cdot\mathbf{K}(\mathbf{x},\mathbf{r}), \qquad \langle\tilde{p}(\mathbf{x})\,\tilde{\boldsymbol{\sigma}}(\mathbf{r})\rangle \equiv -\nabla_{\mathbf{r}}\cdot\mathbf{D}(\mathbf{x},\mathbf{r}) \tag{B.4}$$

then $(\mathbf{K}, \mathbf{D})$ satisfies

$$\left\{R\left(\delta_{ij}\,\bar{\mathbf{v}}\cdot\nabla + \frac{\partial\bar{v}_i}{\partial x_j}\right) - \delta_{ij}\Delta\right\} K_{jklm}(\mathbf{x},\mathbf{r}) + \frac{\partial}{\partial x_i} D_{klm}(\mathbf{x},\mathbf{r}) = \gamma_{iklm}\delta(\mathbf{x}-\mathbf{r}) \tag{B.5}$$

$$\frac{\partial}{\partial x_i} K_{ijlm} = 0$$

Introducing the outer variable $\tilde{\mathbf{x}}$ and expanding in $R_\kappa^{1/2}$ as

$$\begin{aligned}\mathbf{K}(\mathbf{x},\mathbf{r}) &= R_\kappa^{1/2}\mathbf{K}_1(\tilde{\mathbf{x}},\mathbf{r}) + \cdots \\ \mathbf{D}(\mathbf{x},\mathbf{r}) &= R_\kappa \mathbf{D}_1(\tilde{\mathbf{x}},\mathbf{r}) + \cdots\end{aligned} \tag{B.6}$$

and substituting these into (B.5), we obtain

$$\tilde{\Delta}\mathbf{K}_1 - \tilde{\nabla}\mathbf{D}_1 - \mathbf{M}\cdot\mathbf{K}_1 = \gamma\delta(\tilde{\mathbf{x}}-\tilde{\mathbf{r}}), \qquad \tilde{\nabla}\cdot\mathbf{K}_1 = 0 \tag{B.7}$$

to the lowest order in $R_\kappa^{1/2}$, where $\mathbf{M}$ is defined by (D.4). Because Eqs. (B.7) are similar to Eqs. (4.18) for $\mathbf{G}$, we may expect that $\mathbf{K}_1(\tilde{\mathbf{x}},\tilde{\mathbf{r}})$ has the same structure as $\mathbf{G}(\tilde{\mathbf{x}})$, in particular, the same asymptotic behavior. Substituting (B.2), (B.3), and (B.6) into (B.1), we obtain

$$\mathbf{E}(\mathbf{r}) = -R\kappa \lim_{A\to\infty} \int_{S_A} [\mathbf{G}^*(\tilde{\mathbf{x}})\cdot\mathbf{f}_0]\cdot[\nabla_{\tilde{\mathbf{r}}}\mathbf{K}_1(\tilde{\mathbf{x}},\tilde{\mathbf{r}})](\mathbf{C}\cdot\tilde{\mathbf{x}})\cdot d\mathbf{S}(\tilde{\mathbf{x}}) \tag{B.8}$$

to the lowest order. It is shown in ref. 38 that the far-field structure of $\mathbf{G}(\tilde{\mathbf{x}})$ and $\mathbf{G}^*(\tilde{\mathbf{x}})$ is a cubical cone, i.e.,

$$\mathbf{G}(\tilde{\mathbf{x}}) \sim \mathbf{G}^*(\tilde{\mathbf{x}}) \sim |\tilde{\mathbf{x}}|^{-1} \qquad \text{for} \quad \frac{(\tilde{x}_1^2+\tilde{x}_3^2)^{1/2}}{|\tilde{x}_2|^{1/3}} \leqslant O(1) \qquad \text{as} \quad |\tilde{x}_2| \to \infty \tag{B.9}$$

Thus, the dominant contribution from large $|\tilde{\mathbf{x}}|$ to the surface integral (B.8) for fixed $\tilde{\mathbf{r}}$ can be asymptotically estimated as

$$\begin{aligned}&\int_{S_A} (\mathbf{G}^*(\tilde{\mathbf{x}})\cdot\mathbf{f}_0)\cdot[\nabla\mathbf{K}(\tilde{\mathbf{x}},\tilde{\mathbf{r}})](\mathbf{C}\cdot\tilde{\mathbf{x}})\cdot d\mathbf{S}(\tilde{\mathbf{x}}) \\ &\qquad \sim \frac{1}{|\tilde{\mathbf{x}}|}\frac{1}{|\tilde{\mathbf{x}}-\tilde{\mathbf{r}}|^2}|\tilde{\mathbf{x}}|\,|\tilde{\mathbf{x}}|^{2/3} \sim |\tilde{\mathbf{x}}|^{-4/3}\end{aligned} \tag{B.10}$$

Letting $A\to\infty$, we obtain the estimation for any fixed $\mathbf{r}$,

$$\mathbf{E}(\mathbf{r}) = o(R_\kappa) \tag{B.11}$$

The other terms in (3.14) can be estimated similarly, and it is easily seen that those are smaller than $\mathbf{E}(\mathbf{r})$.

APPENDIX C. THE GREEN'S FUNCTION AND THE TENSOR H

The computation of the Green's function $\mathbf{G}$ is the same as in ref. 39, but there are typographical errors in the latter results. We show the

Green's function as follows. It is convenient to introduce the Fourier transform in space as

$$\mathbf{G}(\tilde{\mathbf{r}}) = \frac{1}{8\pi^3} \int \mathbf{G}(\mathbf{k}) \exp(-i\mathbf{k}\tilde{\mathbf{r}})\, d\tilde{\mathbf{r}}$$

The Fourier transform of (4.18) is given by

$$\left(k_1 \frac{\partial}{\partial k_3} - k_3 \frac{\partial}{\partial k_1}\right) G_{ij} - \left(\delta_{i1} - 2\frac{k_i k_1}{k^2}\right) G_{3j} + \left(\delta_{i3} - 2\frac{k_i k_3}{k^2}\right) G_{1j} - k^2 G_{ij} = -P_{ij}(\mathbf{k}) \tag{C.1}$$

where T_j is eliminated by using the solenoidal condition and $P_{ij}(\mathbf{k}) = \delta_{ij} - k_i k_j / k^2$. Introducing the variable $k_1 = -\hat{k} \sin\eta$ and $k_3 = \hat{k} \cos\eta$, and after tedious but straightforward calculation, we obtain $G_{ij}(\mathbf{k})$ as

$$G_{11} = \frac{1}{2}(k^2 \cos^2\eta + k_2^2 \sin^2\eta)\left(\frac{1}{K_+} + \frac{1}{K_-}\right) + \frac{kk_2}{2}\left(\frac{1}{K_-} - \frac{1}{K_+}\right) - \frac{\hat{k}^2}{2k^2} \sin\eta \cos\eta \left[(1+\zeta)\frac{1}{K_+} + (1-\zeta)\frac{1}{K_-}\right] \tag{C.2}$$

$$G_{12} = (k^2 \sin\eta + 2\cos\eta)\frac{k_2 \hat{k}}{k^2(k^4 + \zeta^2)} \tag{C.3}$$

$$G_{13} = \frac{1}{2k^2}(k^2\cos^2\eta + k_2^2 \sin^2\eta)\left[(1+\zeta)\frac{1}{K_+} + (1-\zeta)\frac{1}{K_-}\right] + \frac{k_2}{2k}\left[(1-\zeta)\frac{1}{K_-} - (1+\zeta)\frac{1}{K_+}\right] + \frac{\hat{k}^2}{2}\sin\eta\cos\eta\left(\frac{1}{K_+} + \frac{1}{K_-}\right) \tag{C.4}$$

$$G_{21} = -\frac{k_2\hat{k}}{k^4+1}(k^2\cos\eta - \sin\eta)\left[\left(1+\frac{k_2}{k}\right)\frac{1}{K_-} + \left(1-\frac{k_2}{k}\right)\frac{1}{K_+}\right] + \frac{k_2\hat{k}}{k^2(k^4+1)}(k^2\sin\eta + \cos\eta)\left[(1-\zeta)\left(1+\frac{k_2}{k}\right)\frac{1}{K_-} + (1+\zeta)\left(1-\frac{k_2}{k}\right)\frac{1}{K_+} + 1\right] \tag{C.5}$$

$$G_{22} = \frac{k^2\hat{k}^2}{k^6 + 4k_2^2} \tag{C.6}$$

$$G_{23} = -\frac{k_2\hat{k}}{k^4+1}(k^2 \sin\eta + \cos\eta)\left[\left(1+\frac{k_2}{k}\right)\frac{1}{K_-} + \left(1-\frac{k_2}{k}\right)\frac{1}{K_+}\right]$$
$$-\frac{k_2\hat{k}}{k^2(k^4+1)}(k^2\cos\eta - \sin\eta)\left[(1-\zeta)\left(1+\frac{k_2}{k}\right)\frac{1}{K_-}\right.$$
$$\left.+(1+\zeta)\left(1-\frac{k_2}{k}\right)\frac{1}{K_+}+1\right] \quad \text{(C.7)}$$

$$G_{21} = -\frac{1}{2k^2}(k^2\sin^2\eta + k_2^2\cos^2\eta)\left[(1+\zeta)\frac{1}{K_+} + (1-\zeta)\frac{1}{K_-}\right]$$
$$-\frac{k_2}{2k}\left[(1-\zeta)\frac{1}{K_-} - (1+\zeta)\frac{1}{K_+}\right] + \frac{\hat{k}^2}{2}\sin\eta\cos\eta\left(\frac{1}{K_+}+\frac{1}{K_-}\right) \quad \text{(C.8)}$$

$$G_{32} = -(k^2\cos\eta - 2\sin\eta)\frac{k_2\hat{k}}{k^2(k^4+\zeta^2)} \quad \text{(C.9)}$$

$$G_{33} = \frac{1}{2}(k^2\sin^2\eta + k_2^2\cos^2\eta)\left(\frac{1}{K_+}+\frac{1}{K_-}\right) + \frac{kk_2}{2}\left(\frac{1}{K_-}-\frac{1}{K_+}\right)$$
$$+\frac{\hat{k}^2}{2k^2}\sin\eta\cos\eta\left[(1-\zeta)\frac{1}{K_-} + (1+\zeta)\frac{1}{K_+}\right] \quad \text{(C.10)}$$

where

$$\zeta = 2\frac{k_2}{k}, \qquad K_\pm = k^4 + (1\pm\zeta)^2$$

Also the Stokeslet $s_{ij}(\mathbf{k})$ is of the form

$$s_{ij} = \frac{1}{k^2}P_{ij}(\mathbf{k}) \quad \text{(C.11)}$$

From (4.17) and (4.20), the tensor $\mathbf{H}$ can be written as

$$\mathbf{H} = -\frac{1}{8\pi^3}\int\{\mathbf{G}(\mathbf{k}) - \mathbf{s}(\mathbf{k})\}\,d\mathbf{k} \quad \text{(C.12)}$$

This integral can be carried out analytically by using the spherical coordinates (k, δ, η), where $\hat{k}^2 = k_1^2 + k_3^2$, $\hat{k} = k\cos\delta$, and $k_2 = k\sin\delta$. The components H_{12}, H_{21}, H_{23}, and H_{32} vanish because of the symmetry of G_{ij} and S_{ij}, and thus we obtain (8.6) and (8.7).

APPENDIX D. THE RECIPROCAL RELATION BETWEEN G AND G*

We shall show the reciprocal relation between $\mathbf{G}(\mathbf{x}, \mathbf{r})$ and $\mathbf{G}^*(\mathbf{x}, \mathbf{r})$. Let us write Eqs. (4.18) and (5.16) as

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}[\mathbf{G}, \mathrm{T}] - \mathbf{M} \cdot \mathbf{G} = -\mathbf{I}\delta(\mathbf{x} - \mathbf{r}) \tag{D.1a}$$

$$\nabla \cdot \mathbf{G} = 0 \tag{D.1b}$$

$$\nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}[\mathbf{G}^*, \mathrm{T}^*] - \mathbf{M}^* \cdot \mathbf{G}^* = -\mathbf{I}\delta(\mathbf{x} - \mathbf{r}') \tag{D.2a}$$

$$\nabla \cdot \mathbf{G}^* = 0 \tag{D.2b}$$

where

$$\boldsymbol{\tau}[\mathbf{G}, \mathrm{T}] \equiv -\mathrm{T}\mathbf{I} + \nabla_{\mathbf{x}} \mathbf{G} + (\nabla_{\mathbf{x}} \mathbf{G})^t \tag{D.3}$$

$$\mathbf{M} \equiv (\mathbf{C} \cdot \mathbf{x}) \cdot \nabla_{\mathbf{x}} \mathbf{I} + \mathbf{C}, \qquad \mathbf{M}^* \equiv -(\mathbf{C} \cdot \mathbf{x}) \cdot \nabla_{\mathbf{x}} \mathbf{I} + \mathbf{C}^t \tag{D.4}$$

and the tilde of the argument in $\mathbf{G}$ and $\mathbf{G}^*$ is omitted for simplicity. Now take the scalar product of (D.1a) with $\mathbf{G}^*(\mathbf{x}, \mathbf{r}')$, and (D.2a) with $\mathbf{G}(\mathbf{x}, \mathbf{r})$ and subtract, to obtain

$$\begin{aligned} &\mathbf{G}^*(\mathbf{x}, \mathbf{r}') \cdot \{\nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}[\mathbf{G}, \mathrm{T}] - \mathbf{M} \cdot \mathbf{G}(\mathbf{x}, \mathbf{r})\} \\ &\quad - \mathbf{G}(\mathbf{x}, \mathbf{r}) \cdot \{\nabla_{\mathbf{x}} \cdot \boldsymbol{\tau}[\mathbf{G}^*, \mathrm{T}^*] - \mathbf{M}^* \cdot \mathbf{G}^*(\mathbf{x}, \mathbf{r}')\} \\ &= -(\mathbf{G}^*)^t(\mathbf{x}, \mathbf{r}')\, \delta(\mathbf{x} - \mathbf{r}) + \mathbf{G}(\mathbf{x}, \mathbf{r})\, \delta(\mathbf{x} - \mathbf{r}') \end{aligned} \tag{D.5}$$

Substituting (D.3) and (D.4) into (D.5) and using the relation[32]

$$(\nabla_{\mathbf{x}} \cdot \mathbf{G}^*) \cdot \boldsymbol{\tau}[\mathbf{G}, \mathrm{T}] = \boldsymbol{\tau}[\mathbf{G}^*, \mathrm{T}^*] \cdot (\nabla_{\mathbf{x}} \cdot \mathbf{G}) \tag{D.6}$$

we obtain

$$\begin{aligned} &\nabla_{\mathbf{x}} \cdot \{\mathbf{G}^* \cdot \boldsymbol{\tau}[\mathbf{G}, \mathrm{T}] - \mathbf{G} \cdot \boldsymbol{\tau}[\mathbf{G}^*, \mathrm{T}^*] - (\mathbf{G}^* \cdot \mathbf{C} \cdot \mathbf{x})\mathbf{G}\} \\ &\quad = \mathbf{G}(\mathbf{x}, \mathbf{r})\, \delta(\mathbf{x} - \mathbf{r}') - (\mathbf{G}^*)^t(\mathbf{x}, \mathbf{r}')\, \delta(\mathbf{x} - \mathbf{r}) \end{aligned} \tag{D.7}$$

Integrating this over a sphere of radius K and using Gauss' theorem yields the reciprocal relation as

$$G_{ij}(\mathbf{r}', \mathbf{r}) = G_{ji}^*(\mathbf{r}, \mathbf{r}') \tag{D.8}$$

provided that the surface integral vanishes as $K \to \infty$. From (4.17), (4.20), (5.15), and (5.18), $\mathbf{H}$ and $\mathbf{H}^*$ are defined as

$$\mathbf{H} = -\lim_{\mathbf{r} \to 0} \{\mathbf{G}(\mathbf{r}, 0) - \mathbf{s}(\mathbf{r}, 0)\} \tag{D.9}$$

$$\mathbf{H}^* = -\lim_{\mathbf{r} \to 0} \{\mathbf{G}^*(\mathbf{r}, 0) - \mathbf{s}(\mathbf{r}, 0)\} \tag{D.10}$$

Because of the relation $s_{ij}(\mathbf{r}_1, \mathbf{r}_2) = s_{ji}(\mathbf{r}_2, \mathbf{r}_1)$ for the Stokeslet, the following identity holds:

$$G_{ij}(\mathbf{r}_1, \mathbf{r}_2) - s_{ij}(\mathbf{r}_1, \mathbf{r}_2) = G_{ji}^*(\mathbf{r}_2, \mathbf{r}_1) - s_{ji}(\mathbf{r}_2, \mathbf{r}_1) \tag{D.11}$$

Putting $\mathbf{r}_2 = 0$ and letting $\mathbf{r}_1 \to 0$ in (D.11), and from (D.9) and (D.10), we finally obtain

$$H_{ij}^* = H_{ji} \tag{D.12}$$

This relation can also be shown by the direct calculation of $\mathbf{H}^*$ using the same procedure as in Appendix C.

APPENDIX E. DERIVATION OF THE FIRST-ORDER OUTER FIELD $(\mathbf{W}_1, \Pi_1)$

Using (5.9), we can write (6.8) as

$$\Delta \mathbf{W}_1 - \nabla(\Pi_1 + \mathbf{P}_1^*) - \mathbf{M} \cdot \mathbf{W}_1 = \mathbf{M}^* \cdot \mathbf{Q}_1^*, \qquad \nabla \cdot \mathbf{W}_1 = 0 \tag{E.1}$$

where the tilde is omitted, and the boundary conditions are given by (6.9) and (6.11). If we write the solution of (E.1) as

$$\begin{aligned} \mathbf{W}_1(\mathbf{r}) &= -\tfrac{1}{2}\{\mathbf{G}^*(\mathbf{r}) - \mathbf{G}(\mathbf{r})\} \cdot \mathbf{f}_0 + \mathbf{A}(\mathbf{r}) \\ \Pi_1(\mathbf{r}) + \mathbf{P}_1^*(\mathbf{r}) &= -\tfrac{1}{2}\{\mathbf{T}^*(\mathbf{r}) - \mathbf{T}(\mathbf{r})\} \cdot \mathbf{f}_0 + \mathbf{B}(\mathbf{r}) \end{aligned} \tag{E.2}$$

then $(\mathbf{A}, \mathbf{B})$ must satisfy the following equation:

$$\Delta \mathbf{A} - \nabla \mathbf{B} - \mathbf{M} \cdot \mathbf{A} = -\tfrac{1}{2}(\mathbf{M} + \mathbf{M}^*) \cdot \mathbf{G}^* \cdot \mathbf{f}_0 \tag{E.3a}$$

$$\nabla \cdot \mathbf{A} = 0 \tag{E.3b}$$

and the boundary conditions

$$\begin{aligned} \mathbf{A}(\mathbf{r}) &= o(r^{-1}) \qquad \text{as} \quad |\mathbf{r}| \to 0 \\ &\to 0 \qquad\qquad \text{as} \quad |\mathbf{r}| \to \infty \end{aligned} \tag{E.4}$$

Note that the parentheses on the right-hand side in (E.2) satisfies the matching conditions (6.11). From the definitions of $\mathbf{M}$ and $\mathbf{M}^*$ in

Appendix D and the fact $\mathbf{C} = -\mathbf{C}'$ for rigid rotation, it follows that the right-hand side of (E.3a) vanishes. Thus, we have

$$\mathbf{A}(\mathbf{r}) = 0, \qquad \mathbf{B}(\mathbf{r}) = 0$$

which leads to (6.12).

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